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# Spherically Symmetric Flow of the Compressible Euler Equations

For the Case Including the Origin

By

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**Abstract**

We study the Euler equations of compressible isentropic gas dynamics with spherical symmetry. Due to the presence of the singularity at the origin, little is also known in the case including the origin. In this article, we prove the existence of local solutions for the case including the origin. We construct the approximate solutions by using the method in [6].

## 1 Introduction

We study the Euler equations of compressible isentropic gas dynamics with spherical symmetry. This is governed by the equations

$$\begin{cases} \rho_t + m_x = -\frac{2}{x}m, \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x = -\frac{2}{x}\frac{m^2}{\rho}, \ p(\rho) = \rho^\gamma/\gamma, \ \vec{x} \in \mathrm{R}^3, \ x = |\vec{x}| \geq 0, \end{cases} \tag{1.1}$$

where the scalar functions $\rho(x,t)$, $m(x,t)$ and $p(x,t)$, are the density, the momentum and the pressure of the gas, respectively. On the non-vacuum state $\rho > 0$, $u = m/\rho$ is the velocity. $\gamma \in (1, 5/3]$ is the adiabatic exponent.

Consider the initial boundary value problem (1.1),

$$(\rho, m)|_{t=0} = (\rho_0(x), m_0(x)) \tag{1.2}$$

and

$$m|_{x=0} = 0. \tag{1.3}$$

Observing (1.1), these equations have singularity at the origin. Therefore, little is known in the case including the origin. The only global existence theorem with large $L^\infty$ initial data satisfying

$$0 \leq \{\rho_0(x)\}^\theta/\theta \leq u_0(x) \tag{1.4}$$

was obtained in [1]. On the other hand, for the case outside the origin ($x \geq 1$), the local existence of weak $L^\infty$ solutions was obtained in [6]. The global existence of solutions with large initial data in $L^\infty$ was discussed in [4]. However this result is wrong. Therefore, in this case, no global existence theorem has obtained in general.

In this article, we consider the initial boundary value problem (1.1)-(1.3) for the case including the origin and initial data which don't necessarily satisfy (1.4). At the present time, it is not even clear in which functional space one should work, in order to prove a general existence theorem. Therefore, although the above results are considered in $L^\infty$, we shall work in another functional space.

Our main theorem is as follows.

**Theorem 1.1** *Assume that the initial data are of the form*

$$(\rho, m)|_{t=0} = (\rho_0(x), m_0(x)) = (\tilde{\rho}_0(x)x^{\frac{2}{\gamma-1}}, \tilde{m}_0(x)x^{\frac{\gamma+1}{\gamma-1}})$$

*satisfying*

$$0 \leq \tilde{\rho}_0(x) \leq C_0, \quad \left|\frac{\tilde{m}_0(x)}{\tilde{\rho}_0(x)}\right| \leq C_0,$$

*for some $C_0 > 0$. Then, there exists a local weak solution $(\rho(x,t), m(x,t)) = (\tilde{\rho}(x)x^{\frac{2}{\gamma-1}}, \tilde{m}(x)x^{\frac{\gamma+1}{\gamma-1}})$ of the initial boundary value problem (1)-(3) satisfying*

$$0 \leq \tilde{\rho}(x,t) \leq C(T), \quad \left|\frac{\tilde{m}(x,t)}{\tilde{\rho}(x,t)}\right| \leq C(T),$$

*for some $C(T) \geq C_0$ in the region $\mathbf{R}_+ \times [0,T]$ for some $T \in (0,\infty)$.*

We first transform (1.1). Set $\rho = \tilde{\rho}x^{\frac{2}{\gamma-1}}$, $m = \tilde{m}x^{\frac{\gamma+1}{\gamma-1}}$ and $\xi = \log x$. Then (1.1) becomes

$$\begin{cases} \tilde{\rho}_t + \tilde{m}_\xi = -a_1\tilde{m}, \\ \tilde{m}_t + \left(\frac{\tilde{m}^2}{\tilde{\rho}} + p(\tilde{\rho})\right)_\xi = -a_2\frac{\tilde{m}^2}{\tilde{\rho}} - a_3 p(\tilde{\rho}), \quad p(\tilde{\rho}) = \tilde{\rho}^\gamma/\gamma, \end{cases} \tag{1.5}$$

where $\theta = \frac{\gamma-1}{2}$, $a_1 = \theta^{-1} + 3$, $a_2 = \theta^{-1} + 4$ and $a_3 = \theta^{-1} + 2$.

Our virtual goal is to prove the local existence of solutions to Cauchy problem (1.5) and

$$(\tilde{\rho}, \tilde{m})|_{t=0} = (\tilde{\rho}_0(x), \tilde{m}_0(x)),$$

where $(\tilde{\rho}_0(x), \tilde{u}_0(x) \equiv \tilde{m}_0(x)/\tilde{\rho}_0(x)) \in L^\infty(\mathrm{R})$.

For simplicity, by changing $\xi$ to $x$, $\tilde{\rho}$ to $\rho$ and $\tilde{m}$ to $m$, we have

$$\begin{cases} \rho_t + m_x = -a_1 m, \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x = -a_2 \frac{m^2}{\rho} - a_3 p(\rho), \quad p(\rho) = \rho^\gamma/\gamma. \end{cases} \tag{1.6}$$

This equation can be written as

$$\begin{cases} v_t + f(v)_x = -g(v), \quad x \in \mathbf{R}, \\ v|_{t=0} = v_0(x), \quad v_0 \in L^\infty(\mathbf{R}). \end{cases} \tag{1.7}$$

# 2 Preliminary

In this section, we first review some results of Riemann solutions for the homogeneous system of gas dynamics. Consider the homogeneous system

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x = 0, \quad p(\rho) = \rho^\gamma/\gamma. \end{cases} \tag{2.1}$$

The eigenvalues of the system are

$$\lambda_1 = \frac{m}{\rho} - c, \quad \lambda_2 = \frac{m}{\rho} + c.$$

Any discontinuity in the weak solutions to (2.1) must satisfy the Rankine-Hugoniot condition

$$\sigma(v - v_0) = f(v) - f(v_0),$$

where $\sigma$ is the propagation speed of the discontinuity, $v_0 = (\rho_0, m_0)$ and $v = (\rho, m)$ are the corresponding left state and right state. This means that

$$\begin{cases} m - m_0 = \frac{m_0}{\rho_0}(\rho - \rho_0) \pm \sqrt{\frac{\rho}{\rho_0} \frac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}(\rho - \rho_0), \\ \sigma = \frac{m - m_0}{\rho - \rho_0} = \frac{m_0}{\rho_0} \pm \sqrt{\frac{\rho}{\rho_0} \frac{p(\rho) - p(\rho_0)}{\rho - \rho_0}}. \end{cases}$$

A discontinuity is called a shock if it satisfies the entropy condition

$$\sigma(\eta(v) - \eta(v_0)) - (q(v) - q(v_0)) \geq 0$$

for any convex entropy pair $(\eta, q)$.

Consider the Riemann problem of (2.1) with initial data

$$v|_{t=0} = \begin{cases} v_-, & x < x_0, \\ v_+, & x > x_0, \end{cases} \tag{2.2}$$

where $x_0 \in (-\infty, \infty)$, $\rho_\pm \geq 0$ and $m_\pm$ are constants satisfying $|m_\pm/\rho_\pm| < \infty$.

Then the following lemmas hold.

**Lemma 2.1** *There exists a unique piecewise entropy solution $(\rho(x,t), m(x,t))$ containing the vacuum state ($\rho = 0$) on the upper plane $t > 0$ for the problem of* (2.2) *satisfying,*

$$\begin{cases} w(\rho(x,t), m(x,t)) \leq \max(w(\rho_-, m_-), w(\rho_+, m_+)), \\ z(\rho(x,t), m(x,t)) \geq \min(z(\rho_-, m_-), z(\rho_+, m_+)), \\ w(\rho(x,t), m(x,t)) - z(\rho(x,t), m(x,t)) \geq 0. \end{cases}$$

Such solutions have the following properties.

**Lemma 2.2** *The regions $\sum = \{(\rho, m) : w \leq w_0, z \geq z_0, w - z \geq 0\}$ are invariant with respect to both of the Riemann problem* (2.2) *and the average of the Riemann solutions in $x$. More preciously, if the Riemann date lie in $\sum$, the corresponding Riemann solutions $(\rho(x,t), m(x,t))$ lie in $\sum$, and their corresponding averages in $x$ also in $\sum$*

$$\left(\frac{1}{b-a}\int_a^b \rho(x,t)dx, \frac{1}{b-a}\int_a^b m(x,t)dx\right) \in \sum.$$

The proof of Lemma 2.2 can be found in [2].

# 3 Approximate Solutions

In this section we construct approximate solutions $v^h = (\rho^h, m^h) = (\rho^h, \rho^h u^h)$ in the strip $0 \leq t \leq T$ for some fixed $T \in (0, \infty)$, where $h$ is the space mesh length, together with the time mesh length $\Delta t$, satisfying the following Courant-Friedrichs-Levy condition

$$2\Lambda \equiv 2 \max_{i=1,2}( \sup_{0 \leq t \leq T} |\lambda_i(\rho^h, m^h)|) \leq \frac{h}{\Delta t} \leq 3\Lambda. \tag{3.1}$$

We will prove that the approximate solutions are bounded uniformly in the mesh length $h > 0$ and $\rho^h(x,t) \geq 0$ to guarantee the construction of $(\rho^h, m^h)$.

We construct the approximate solutions $(\rho^h, m^h)$. Let

$$t_n = n\Delta t, \quad x_j = jh, \quad (n,j) \in \mathbf{Z}_+ \times \mathbf{Z}.$$

Assume that $v^h(x,t)$ is defined for $t < n\Delta t$. Then we define $v_j^n \equiv (\rho_j^n, m_j^n)$ as, for $j \in \mathbf{Z}$,

$$\begin{cases} \rho_j^n \equiv \frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \rho^h(x, n\Delta t - 0)dx, & (j-\frac{1}{2})h \le x \le (j+\frac{1}{2})h, \\ m_j^n \equiv \frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} m^h(x, n\Delta t - 0)dx, & (j-\frac{1}{2})h \le x \le (j+\frac{1}{2})h. \end{cases} \tag{3.2}$$

Then, in the strip $n\Delta t \le t < (n+1)\Delta t$, $v_0^h(x,t)$ is defined as, for $jh \le x < (j+1)h$ ($j \in \mathbf{Z}$), the solution of the Riemann problem at $x = (j+\frac{1}{2})h$

$$\begin{cases} v_t + f(v)_x = 0, \quad jh \le x < (j+1)h, \\ v|_{t=n\Delta t} = \begin{cases} v_j^n, & x < (j+\frac{1}{2})h, \\ v_{j+1}^n, & x > (j+\frac{1}{2})h. \end{cases} \end{cases}$$

Finally we define $v^h(x,t)$ in the strip $n\Delta t \le t < (n+1)\Delta t$ by the fractional step procedure:

$$v^h(x,t) = v_0^h(x,t) + g(v_0^h(x,t))(t - n\Delta t).$$

# 4 $L^\infty$ Estimates

We derive a $L^\infty$ bound for the approximate solutions $v^h(x,t)$ of the initial value problem (1.7).

**Theorem 4.1** *Assume that the initial velocity and nonnegative density data* $(\rho_0, u_0) \in L^\infty(\mathbf{R})$. *Then there exists a* $T > 0$ *such that the difference approximate solutions of the initial value problem* (1.7) *are uniformly bounded. That is, there exists a constant* $C > 0$ *such that*

$$|u^h(x,t)| \le C, \quad 0 \le \rho^h(x,t) \le C, \quad (x,t) \in \mathbf{R} \times [0,T]. \tag{4.1}$$

*Proof.* Set

$$M_n = \max(\sup_x w(v^h(x, n\Delta t + 0)), -\inf_x z(v^h(x, n\Delta t + 0)), 1).$$

For $n\Delta t \le t < (n+1)\Delta t, n \ge 0$ integer, we use Lemma 2.1 and the construction of $(\rho_0^h, m_0^h)$ to get

$$
\begin{aligned}
w(v^h) &= u^h + \frac{(\rho^h)^\theta}{\theta} \\
&= w(v_0^h) - \left\{(u_0^h)^2 + (\theta^{-1}+3)u_0^h(\rho_0^h)^\theta + (\theta^{-1}+2)(\rho_0^h)^{2\theta}/\gamma\right\}(t-n\Delta t) + \mathbf{o}(\Delta t) \\
&\le w(v_0^h) - \frac{1}{4}\left\{(2+4\theta)(w(v_0^h))^2 + 2(1-\theta)w(v_0^h)z(v_0^h) - 2\theta(z(v_0^h))^2\right\}\Delta t + \mathbf{o}(\Delta t) \\
&\le M_n - \frac{1}{4}\left\{(2+4\theta)M_n^2 + 2(1-\theta)M_n z(v_0^h) - 2\theta(z(v_0^h))^2\right\}\Delta t + \mathbf{o}(\Delta t) \\
&\le M_n + \mathbf{o}(\Delta t),
\end{aligned}
$$

and

$$
\begin{aligned}
z(v^h) &= u^h - \frac{(\rho^h)^\theta}{\theta} \\
&= z(v_0^h) - \left\{(u_0^h)^2 - (\theta^{-1}+3)u_0^h(\rho_0^h)^\theta + (\theta^{-1}+2)(\rho_0^h)^{2\theta}/\gamma\right\}(t-n\Delta t) + \mathbf{o}(\Delta t) \\
&= z(v_0^h) - \frac{1}{4}\left\{-2\theta(w(v_0^h))^2 + 2(1-\theta)w(v_0^h)z(v_0^h) + (2+4\theta)(z(v_0^h))^2\right\}\Delta t + \mathbf{o}(\Delta t) \\
&\ge -M_n - \frac{1}{4}\left\{-2\theta(w(v_0^h))^2 - 2(1-\theta)w(v_0^h)M_n + (2+4\theta)M_n^2\right\}\Delta t + \mathbf{o}(\Delta t) \\
&\ge -M_n - M_n^2\Delta t + \mathbf{o}(\Delta t),
\end{aligned}
$$

where Landau symbol $\mathbf{o}(\Delta t)$ is a constant depending only on the uniform bound of $v_0^h$ and $\mathbf{o}(\Delta t)/\Delta t \to 0$, as $\Delta t \to 0$.

Therefore it follows from Lemma 2.2 that

$$M_{n+1} \le M_n(1 + M_n\Delta t),$$

that is,

$$\frac{M_{n+1} - M_n}{\Delta t} \le M_n^2. \tag{4.2}$$

Consider the corresponding ordinary differential equation

$$\begin{cases} \frac{dr}{dt} = r^2, \\ r(0) = r_0. \end{cases} \tag{4.3}$$

It follows that

$$r_0 \le r(t) \le \tilde{C}(T) \equiv \frac{1}{-T + \frac{1}{r_0}}, \quad 0 \le t \le T < \frac{1}{r_0}. \tag{4.4}$$

Noting the integral curve $r = r(t)$ is convex curve, we obtain from (4.2)-(4.4) that

$$M_n \leq r(n\Delta t) \leq \tilde{C}(T). \tag{4.5}$$

Therefore, it follows that (4.5) for $n\Delta t \leq t < (n+1)\Delta t$, that is, there is a constant $C > 0$ such that

$$|u^h(x,t)| = \left|\frac{m^h(x,t)}{\rho^h(x,t)}\right| \leq C, \quad 0 \leq \rho^h(x,t) \leq C,$$

by choosing $\Delta t$ enough small.

The following proposition and theorem can be proved in the same manner to [5] and [6].

**Proposition 4.2** *The measure sequence*

$$\eta(v^h)_t + q(v_h)_x$$

*lies in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$ for all weak entropy pair $(\eta, q)$, where $\Omega \subset \mathbf{R} \times [0,T]$ is any bounded and open set.*

**Theorem 4.3** *Assume that the approximate solution $(\rho^h, m^h)$ satisfy Theorem 4.1 and Proposition 4.2. Then there is a convergent subsequence in the approximate solutions $(\rho^h(x,t), m^h(x,t))$ such that*

$$(\rho^{h_n}(x,t), m^{h_n}(x,t)) \to (\rho(x,t), m(x,t)), \quad a.e. \tag{4.6}$$

*The pair of the functions $(\rho(x,t), m(x,t))$ is a local entropy solution of the initial-boundary value problem (1.7) satisfying*

$$0 \leq \rho(x,t) \leq C, \quad \left|\frac{m(x,t)}{\rho(x,t)}\right| \leq C, \tag{4.7}$$

*for some $C$ in the region $\mathbf{R} \times [0,T]$.*

# 5 Open Problem

Here we list some open problems related to this paper.

- We first introduce a example.

$$\rho(x,t) = \frac{C_2}{(t+C_1)^3}, \quad u(x,t) = \frac{x}{t+C_1}, \tag{5.1}$$

where $C_1$ and $C_2$ are constants. (5.1) is a solution of (1.1). If $C_1 > 0$ (i.e. the initial velocity is positive), this solution is global. On the other hand, if $C_1 < 0$ (i.e. the initial velocity is negative), this solution blows up. Therefore a blow up solution certainly exists. Then can another blow up solution be constructed, perfectly in more general?

- For the case initial Riemann Invariant $z$ is nonnegative, the global existence of solutions has obtained in [1]. Can the global existence of solutions (not necessarily including the origin) be proved except this result (of course, and (5.1))? In addition, since [4] is wrong, notice that the global existence theorem for duct flow and self-gravitating gases isn't also obtained.

- The initial density of [1] and theorem 1.2 is 0 at the origin. Can the existence (not necessarily global) with initial density, which isn't 0 at the origin, be proved (of course, except (5.1))?

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